

Equations for polar Grassmannians

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Abstract

Given an N -dimensional vector space V over a field \mathbb{F} and a trace-valued (σ, ε) -sesquilinear form $f : V \times V \rightarrow \mathbb{F}$, with $\varepsilon = \pm 1$ and $\sigma^2 = \text{id}_{\mathbb{F}}$, let \mathcal{S} be the polar space of totally f -isotropic subspaces of V and let n be the rank of \mathcal{S} . Assuming $n \geq 2$, let $2 \leq k \leq n$, let \mathcal{G}_k the k -grassmannian of $\text{PG}(V)$, embedded in $\text{PG}(\wedge^k V)$ as a projective variety and \mathcal{S}_k the k -grassmannian of \mathcal{S} . In this paper we find one simple equation that, jointly with the equations of \mathcal{G}_k , describe \mathcal{S}_k as a subset of $\text{PG}(\wedge^k V)$.

1 Introduction

Throughout this paper \mathbb{F} is a commutative field, N is a positive integer, $V := V(N, \mathbb{F})$ and $f : V \times V \rightarrow \mathbb{F}$ is a trace-valued (σ, ε) -sesquilinear form on V , with $\varepsilon = \pm 1$ and $\sigma^2 = \text{id}_{\mathbb{F}}$. Let \mathcal{S} be the polar space of totally f -isotropic subspaces of V (Tits [3], Buekenhout and Cohen [1]) and let n be the rank of \mathcal{S} , namely the Witt index of f .

Assuming $n \geq 2$, let $2 \leq k \leq n$. We denote by \mathcal{G}_k the k -grassmannian of $\text{PG}(V)$, embedded in $\text{PG}(\wedge^k V)$ as a projective variety, and by \mathcal{S}_k the k -grassmannian of \mathcal{S} , regarded as a subset of \mathcal{G}_k , whence a subset of $\text{PG}(\wedge^k V)$. In this paper we address the problem of finding equations that, jointly with those of \mathcal{G}_k , describe \mathcal{S}_k .

A partial answer to this problem is given in Cardinali and Pasini [2], where the case $k = 2$ is solved. Explicitly, let M_f be the representative matrix of f . Remarking that $\wedge^2 V$ can be regarded as the space of alternating $N \times N$ matrices over \mathbb{F} , the following is proved in [2]:

Theorem 1.1 *The following matrix equation, jointly with the equations of \mathcal{G}_2 , characterizes \mathcal{S}_2 :*

$$M_f^\sigma X^\sigma M_f X = O \quad (1)$$

where O is the $N \times N$ null matrix and X ranges over the set of alternating $N \times N$ matrices.

When f is non-degenerate equation (1) can be simplified as follows:

$$X^\sigma M_f X = O. \quad (2)$$

In this paper we shall generalize the above result. Note firstly that, for $2 \leq k \leq n$, the vectors of $\wedge^k V$ can be regarded as alternating tensors of degree k :

$$\sum_{i_1 < \dots < i_k} x_{i_1, \dots, i_k} e_{i_1} \wedge \dots \wedge e_{i_k} \longleftrightarrow (y_{i_1, \dots, i_k})_{i_1, \dots, i_k=1}^N,$$

where if $|\{i_1, \dots, i_k\}| = k$ then $y_{i_1, \dots, i_k} = \text{sign}(p) \cdot x_{p(i_1), \dots, p(i_k)}$ for a permutation p of $\{i_1, \dots, i_k\}$ with $p(i_1) < p(i_2) < \dots < p(i_k)$ and $y_{i_1, \dots, i_k} = 0$ when $|\{i_1, \dots, i_k\}| < k$. Needless to say, (e_1, \dots, e_N) is a given basis of V and the vectors $e_{i_1} \wedge \dots \wedge e_{i_k}$ (for $i_1 < \dots < i_k$) form the basis of $\wedge^k V$ naturally associated to it.

The main result of this paper is a tensor equation which includes (1) as a special case and, combined with the equations of \mathcal{G}_k , characterizes \mathcal{S}_k . In order to write our equation, we must recall a few facts from old fashion tensor calculus and fix some notation, suited to our needs.

1.1 Preliminaries from tensor calculus

All tensors to be considered in the sequel belong to $\otimes^r V$ for some r , by assumption. In particular, all matrices are $N \times N$ matrices. If $X \in \otimes^r V$, the integer r is called the *degree* of X . Given two tensors X and Y of degree r and s respectively

$$X = (x_{i_1, \dots, i_r})_{i_1, \dots, i_r=1}^N, \quad Y = (y_{j_1, \dots, j_s})_{j_1, \dots, j_s=1}^N,$$

let $p \leq \min(r, s)$. We put

$$X \overset{p}{\circ} Y := (z_{i_1, \dots, i_{r-p}, j_{p+1}, \dots, j_s})_{i_1, \dots, i_{r-p}, j_{p+1}, \dots, j_s=1}^N$$

where

$$z_{i_1, \dots, i_{r-p}, j_{p+1}, \dots, j_s} := \sum_{h_1, \dots, h_p} x_{i_1, \dots, i_{r-p}, h_1, \dots, h_p} y_{h_1, \dots, h_p, j_{p+1}, \dots, j_s}.$$

The tensor $X \overset{p}{\circ} Y$ has degree $r + s - 2p$. It is called the *p-product* of X and Y . We also write $X \circ Y$ for $X \overset{1}{\circ} Y$, for short. Thus,

$$X \circ Y := \left(\sum_i x_{i_1, \dots, i_{r-1}, i} y_{i, j_2, \dots, j_s} \right)_{i_1, \dots, i_{r-1}, j_2, \dots, j_s=1}^N.$$

When X and Y are $N \times N$ matrices (tensors of degree 2) then $X \circ Y$ is just their usual row-times-column product. If X and Y are vectors (tensors of degree 1) then $X \circ Y$ is their so-called scalar product, namely their row-times-column product with X regarded as a $1 \times N$ matrix and Y as an $N \times 1$ matrix.

Turning back to the general case, the following associative law holds, provided that all products involved in it are defined:

$$(X \overset{p}{\circ} Y) \overset{q}{\circ} Z = X \overset{p}{\circ} (X \overset{q}{\circ} Z). \quad (3)$$

Thus, we are allowed to write $X \overset{p}{\circ} Y \overset{q}{\circ} Z$, omitting parentheses. Moreover, if I is the identity matrix then the following also holds for any tensor X :

$$I \circ X = X \circ I = X. \quad (4)$$

With X and Y as above, the *tensor product* of X and Y is the tensor $X \otimes Y$ of degree $r + s$ defined as follows:

$$X \otimes Y := (x_{i_1, \dots, i_r} y_{j_1, \dots, j_s})_{i_1, \dots, i_r, j_1, \dots, j_s=1}^N.$$

In view of our purposes, we need a slight modification of this definition. Assume that r and s are even, say $r = 2u$ and $s = 2v$. Then we define

$$X \odot Y := (z_{i_1, \dots, i_u, j_1, \dots, j_v, i_{u+1}, \dots, i_r, j_{v+1}, \dots, j_s})_{i_1, \dots, i_r, j_1, \dots, j_s=1}^N,$$

where

$$z_{i_1, \dots, i_u, j_1, \dots, j_v, i_{u+1}, \dots, i_r, j_{v+1}, \dots, j_s} = x_{i_1, \dots, i_r} y_{j_1, \dots, j_s}.$$

We call $X \odot Y$ the *pseudo-tensor product* of X and Y . Note that $X \odot Y$ is the same as $X \otimes Y$ but for a permutation of the indices.

If X is a tensor of even degree (in particular, a matrix), we also define *pseudo-tensor powers* as follows:

$$\odot^1 X := X, \quad \odot^{r+1} X := (\odot^r X) \odot X.$$

It is worth to recall a few properties of products and powers introduced so far. Their proofs are straightforward. We leave them to the reader.

Proposition 1.2 *Let A and B be matrices. Then both the following hold:*

$$(\odot^r A) \odot (\odot^s A) = \odot^{r+s} A, \quad (\odot^r B) \overset{r}{\circ} (\odot^r A) = \odot^r AB, \quad (5)$$

where r and s are positive integers and AB ($= A \circ B$) is the usual row-times-column product of matrices.

Proposition 1.3 *Let A be a matrix, t a positive integer and X a tensor of degree t . Let I be the identity matrix. Then all the following hold:*

$$(\odot^s t I) \overset{s}{\circ} X = X \overset{s}{\circ} (\odot^s I) = X \text{ for any positive integer } s \leq t, \quad (6)$$

$$X \overset{t}{\circ} (\odot^t A) = (\odot^t A^T) \overset{t}{\circ} X = (\odot^{t-1} A^T) \overset{t-1}{\circ} X \circ A. \quad (7)$$

Corollary 1.4 *With A , X and t as in Proposition 1.3, assume that A is non-singular. Then the following holds for any tensor Y :*

$$X \overset{t}{\circ} (\odot^t A) \circ Y = O \implies X \circ A \circ Y = O \quad (8)$$

where O stands for the null tensor of degree $t + s - 2$, s being the degree of Y .

We mention one more property, to be exploited in the proof of Lemma 2.2.

Proposition 1.5 *The following holds for any two tensors X and Y and any two vectors x and y :*

$$(X \otimes x) \circ (y \otimes Y) = (x \circ y) \cdot (X \otimes Y). \quad (9)$$

We leave the proof to the reader. We only warn that $x \circ y$ is a scalar.

1.2 Main result

We are now ready to state our main theorem. We shall prove it in Section 2.

Theorem 1.6 *Let M_f be the representative matrix of f and $2 \leq k \leq n = \text{rank}(\mathcal{S})$. Then the following tensor equation, jointly with the equations of \mathcal{G}_k , characterizes \mathcal{S}_k :*

$$X^\sigma \circ^k (\odot^k M_f) \circ X = O \quad (10)$$

where O is the null tensor of degree $2k-2$, the unknown $X = (x_{i_1, \dots, i_k})_{i_1, \dots, i_k=1}^N$ ranges in the set of alternating tensors of degree k and $X^\sigma := (x_{i_1, \dots, i_k}^\sigma)_{i_1, \dots, i_k=1}^N$.

By (7) of Proposition 1.3 and the equality $M_f^T = \varepsilon M_f^\sigma$, equation (10) can be given the following form, which includes (1) as a special case:

$$(\odot^{k-1} M_f^\sigma) \circ^{k-1} X^\sigma \circ M_f \circ X = O. \quad (11)$$

By (10) and implication (8) of Corollary 1.4 we immediately obtain the following:

Corollary 1.7 *When f is non-degenerate then \mathcal{S}_k is characterized by the following tensor equation (combined with the equations of \mathcal{G}_k):*

$$X^\sigma \circ M_f \circ X = O. \quad (12)$$

Note. We have assumed $k \geq 2$ in Theorem 1.6, but (10) trivially holds when $k = 1$ too, provided that we put $\odot^0 M_f^\sigma := 1$, $1 \circ^0 X^\sigma = X^\sigma$ and take the phrase “alternating tensor of degree 1” as an oddish synonym of “vector”.

2 Proof of Theorem 1.6

Let V^* be the dual of V . We regard V as a right vector space. Accordingly, V^* is a left vector space. We recall that the vectors of V^* are linear functionals $\xi : V \rightarrow \mathbb{F}$.

Given a basis $E = (e_i)_{i=1}^N$ of V let $E^* = (e_i^*)_{i=1}^N$ be the basis of V^* dual to E . Thus, $e_j^*(e_i) = \delta_{i,j}$ (Kronecker symbol) for $i, j = 1, 2, \dots, N$. We take the $\binom{N}{k}$ -tuples

$$\begin{aligned} \wedge^k E &= (e_{i_1} \wedge \dots \wedge e_{i_k})_{1 \leq i_1 < \dots < i_k \leq N}, \\ \wedge^k E^* &= (e_{i_1}^* \wedge \dots \wedge e_{i_k}^*)_{1 \leq i_1 < \dots < i_k \leq N} \end{aligned}$$

as bases of $\wedge^k V$ and $\wedge^k V^*$ respectively. Given k independent vectors x_1, \dots, x_k of V , for $r = 1, 2, \dots, k$ let $x_{1,r}, \dots, x_{N,r}$ be the coordinates of x_r with respect to E . Then

$$x_1 \wedge x_2 \wedge \dots \wedge x_k = \sum_{i_1 < i_2 < \dots < i_k} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \cdot x_{i_1, i_2, \dots, i_k}$$

where $x_{i_1, \dots, i_k} = \det(x_{i_r, i})_{r, i=1}^k$. We denote by X_E the alternating tensor corresponding to $x_1 \wedge \dots \wedge x_k$, the subscript E being a reminder of the basis E chosen in V . Explicitly,

$$X_E = (y_{i_{p(1)}, \dots, i_{p(k)}} \mid p : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, k\}, i_1 < i_2 < \dots < i_k)$$

where $y_{i_{p(1)}, \dots, i_{p(k)}} = \text{sign}(p) \cdot x_{i_1, \dots, i_k}$ if p is a permutation and $y_{i_{p(1)}, \dots, i_{p(k)}} = 0$ if the mapping p is non-injective.

Similarly, given an independent k -tuple (ξ_1, \dots, ξ_k) in V^* , for $s = 1, 2, \dots, k$ let $\xi_{s,1}, \dots, \xi_{s,N}$ be the coordinates of ξ_s with respect to E . Then

$$\xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_k = \sum_{j_1 < j_2 < \dots < j_k} \xi_{j_1, j_2, \dots, j_k} \cdot e_{j_1}^* \wedge e_{j_2}^* \wedge \dots \wedge e_{j_k}^*$$

where $\xi_{j_1, \dots, j_k} = \det(\xi_{j, j_s})_{j, s=1}^k$. The alternating tensor corresponding to the vector $\xi_1 \wedge \dots \wedge \xi_k$ will be denoted by Ξ_E .

Given another basis $F = (f_1, \dots, f_N)$ of V let $F^* = (f_1^*, \dots, f_N^*)$ be the basis of V^* dual of F . With x_1, \dots, x_k and ξ_1, \dots, ξ_k as above. let X_F and Ξ_F be the tensors representing $x_1 \wedge \dots \wedge x_k$ and $\xi_1 \wedge \dots \wedge \xi_k$ with respect to the choice of F as the basis of V .

Lemma 2.1 *We have $\Xi_F \circ X_F = \Xi_E \circ X_E$, where \circ is the 1-product defined in Section 1.1.*

Proof. Let C be the matrix mapping E onto F . Then C^{-1} maps E^* onto F^* . Explicitly, if $C = (c_{i,j})_{i,j=1}^N$ and $C^{-1} = (c'_{i,j})_{i,j=1}^N$, then $f_r = \sum_{i=1}^N e_i c_{i,r}$ and $f_r^* = \sum_{j=1}^N c'_{r,j} e_j^*$ for $r = 1, 2, \dots, N$. Consequently, the tensors X_E and Ξ_E representing $x_1 \wedge \dots \wedge x_k$ and $\xi_1 \wedge \dots \wedge \xi_k$ are changed to $X_F := (\odot^k C^{-1}) \circ^k X_E$ and $\Xi_F := \Xi_E \circ^k (\odot^k C)$. Thus

$$\Xi_F \circ X_F = \left(\Xi_E \circ^k (\odot^k C) \right) \circ \left(\odot^k C^{-1} \right) \circ^k X_E. \quad (13)$$

By repeated applications of (7) of Proposition 1.3, associativity and the second equation of (5) of Proposition 1.2 we get

$$\begin{aligned} & \left(\Xi_E \circ^k (\odot^k C) \right) \circ \left(\odot^k C^{-1} \right) \circ^k X_E = \\ & = \left((\odot^{k-1} C^T) \circ^{k-1} (\odot^{k-1} C^{-T}) \right) \circ^{k-1} (\Xi_E \circ C \circ C^{-1} \circ X_E) = \\ & = (\odot^{k-1} (C^T \circ C^{-T})) \circ^{k-1} (\Xi_E \circ C \circ C^{-1} \circ X_E). \end{aligned}$$

However $C^T \circ C^{-T} = C^T C^{-T} = I$ and $C \circ C^{-1} = C C^{-1} = I$. Therefore

$$\left(\Xi_E \circ^k (\odot^k C) \right) \circ \left(\odot^k C^{-1} \right) \circ^k X_E = \Xi_E \circ X_E \quad (14)$$

by (6) of Proposition 1.3. The lemma now follows from (13) and (14). \square

Henceforth we write X and Ξ for short instead of X_E and Ξ_E .

Lemma 2.2 *We have $\langle x_1, \dots, x_k \rangle \subseteq \cap_{i=1}^k \text{Ker}(\xi_i)$ if and only if $\Xi \circ X = O$, where O stands for the null tensor of degree $2k - 2$.*

Proof. Let $(x_{i_r, i})_{r, i=1}^k$ and $(\xi_{j, j_s})_{j, s=1}^k$ be the matrices introduced at the beginning of Section 2, when describing $x_1 \wedge \dots \wedge x_k$ and $\xi_1 \wedge \dots \wedge \xi_k$. Recall that

$$\det(x_{i_r, i})_{r, i=1}^k = \sum_{\sigma \in \text{Sym}(k)} \text{sign}(\sigma) \prod_{r=1}^k x_{i_r, \sigma(r)}.$$

Moreover, given a cyclic permutation γ of $\{1, 2, \dots, k\}$ every permutation of $\{1, 2, \dots, k\}$ splits as the product of a power γ^u of γ and a permutation of $\gamma^u(\{1, 2, \dots, k-1\})$ as well as the product of a power γ^v of γ and a permutation of $\gamma^v(\{2, 3, \dots, k\})$. Therefore

$$\begin{aligned} \det(x_{i_r, i})_{r, i=1}^k &= \sum_{u=0}^{k-1} (-1)^{(k-1)u} \det(x_{i_r, \gamma^u(i)})_{r, i=1}^{k-1} \cdot x_{i_k, \gamma^u(k)} = \\ &= \sum_{v=0}^{k-1} (-1)^{(k-1)v} x_{i_1, \gamma^v(1)} \cdot \det(x_{i_r, \gamma^v(i)})_{r, i=2}^k. \end{aligned}$$

It follows that

$$\begin{aligned} X &= \sum_{u=0}^{k-1} (-1)^{(k-1)u} X(x_{\gamma^u(1)}, \dots, x_{\gamma^u(k-1)}) \otimes x_{\gamma^u(k)} = \\ &= \sum_{v=0}^{k-1} (-1)^{(k-1)v} x_{\gamma^v(1)} \otimes X(x_{\gamma^v(2)}, \dots, x_{\gamma^v(k)}). \end{aligned}$$

where $X(x_{\gamma^u(1)}, \dots, x_{\gamma^u(k-1)})$ is the tensor corresponding to $x_{\gamma^u(1)} \wedge \dots \wedge x_{\gamma^u(k-1)} \in \wedge^{k-1} V$ and $X(x_{\gamma^v(2)}, \dots, x_{\gamma^v(k)})$ corresponds to $x_{\gamma^v(2)} \wedge \dots \wedge x_{\gamma^v(k)}$. Similarly,

$$\begin{aligned} \Xi &= \sum_{u=0}^{k-1} (-1)^{(k-1)u} \Xi(\xi_{\gamma^u(1)}, \dots, \xi_{\gamma^u(k-1)}) \otimes \xi_{\gamma^u(k)} = \\ &= \sum_{v=0}^{k-1} (-1)^{(k-1)v} \xi_{\gamma^v(1)} \otimes \Xi(\xi_{\gamma^v(2)}, \dots, \xi_{\gamma^v(k)}). \end{aligned}$$

Therefore

$$\begin{aligned} \Xi \circ X &= \left(\sum_{u=0}^{k-1} (-1)^{(k-1)u} \Xi(\xi_{\gamma^u(1)}, \dots, \xi_{\gamma^u(k-1)}) \otimes \xi_{\gamma^u(k)} \right) \circ \\ &\quad \circ \left(\sum_{v=0}^{k-1} (-1)^{(k-1)v} x_{\gamma^v(1)} \otimes X(x_{\gamma^v(2)}, \dots, x_{\gamma^v(k)}) \right). \end{aligned}$$

By the above and (9) of Proposition 1.5 we immediately obtain the following:

$$\begin{aligned} \Xi \circ X &= \sum_{u,v=0}^{k-1} (-1)^{(k-1)(u+v)} \xi_{\gamma^v(k)}(x_{\gamma^u(1)}) \cdot \\ &\quad \cdot (\Xi(\xi_{\gamma^u(1)}, \dots, \xi_{\gamma^u(k-1)}) \otimes X(x_{\gamma^u(2)}, \dots, x_{\gamma^u(k)})) . \end{aligned} \quad (15)$$

If $\langle x_1, \dots, x_k \rangle \subseteq \cap_{i=1}^k \text{Ker}(\xi_i)$ then $\xi_{\gamma^v(k)}(x_{\gamma^u(1)}) = 0$ for any choice of $u, v = 0, 1, \dots, k-1$. Therefore $\Xi \circ X = O$ by (15). The ‘only if’ part of the lemma is proved.

Turning to the ‘if’ part, let $\Xi \circ X = O$. In view of Lemma 2.1 we may assume to have chosen the basis E in such a way that $e_i = x_i$ for $i = 1, 2, \dots, k$. Thus,

$$X = \sum_{v=0}^{k-1} (-1)^{(k-1)v} e_{\gamma^v(1)} \otimes X(e_{\gamma^v(2)}, \dots, e_{\gamma^v(k)}).$$

We can now rewrite the hypothesis $\Xi \circ X = O$ as follows:

$$\begin{aligned} O &= \sum_{u,v=0}^{k-1} (-1)^{(k-1)(u+v)} \xi_{\gamma^v(k)}(e_{\gamma^u(1)}) \cdot \\ &\quad \cdot (\Xi(\xi_{\gamma^u(1)}, \dots, \xi_{\gamma^u(k-1)}) \otimes X(e_{\gamma^u(2)}, \dots, e_{\gamma^u(k)})) . \end{aligned} \quad (16)$$

The tensors $X(e_2, \dots, e_k), X(e_{\gamma(2)}, \dots, e_{\gamma(k)}), \dots, X(e_{\gamma^{k-1}(2)}, \dots, e_{\gamma^{k-1}(k)})$ are linearly independent. Hence (16) yields

$$\sum_{v=0}^{k-1} (-1)^{(k-1)(u+v)} \xi_{\gamma^v(k)}(e_{\gamma^u(1)}) \cdot \Xi(\xi_{\gamma^u(1)}, \dots, \xi_{\gamma^u(k-1)}) = O \quad (17)$$

for $u = 0, 1, \dots, k-1$, where O now stands for the null tensor of degree $k-1$. In order to prove that $\langle e_1, \dots, e_k \rangle \subseteq \cap_{i=1}^k \text{Ker}(\xi_i)$ we must show that $\xi_j(e_i) = 0$ for any choice of $i, j = 1, 2, \dots, k$.

Suppose the contrary. Let $\xi_k(e_1) \neq 0$, to fix ideas. If we replace the k -tuple (ξ_1, \dots, ξ_k) with another k -tuple (ξ'_1, \dots, ξ'_k) such that $\langle \xi_1, \dots, \xi_k \rangle = \langle \xi'_1, \dots, \xi'_k \rangle$ then Ξ is changed to $\Xi' = \lambda \Xi$ for a scalar $\lambda \neq 0$. So, we can assume to have chosen ξ_1, \dots, ξ_k in such a way that $\xi_k(e_1) = 1$ and $\xi_j(e_1) = 0$ for $j < k$. Namely, $\xi_{\gamma^0(k)}(e_1) = 1$ and $\xi_{\gamma^v(k)}(e_1) = 0$ for $v > 0$. Therefore

$$\sum_{v=0}^{k-1} (-1)^{(k-1)v} \xi_{\gamma^v(k)}(e_1) \cdot \Xi(\xi_{\gamma^u(1)}, \dots, \xi_{\gamma^u(k-1)}) = \Xi(\xi_1, \dots, \xi_{k-1}). \quad (18)$$

By (18) and (17) with $u = 0$ we obtain $\Xi(\xi_1, \dots, \xi_{k-1}) = O$. However this is impossible. Indeed $\Xi(\xi_1, \dots, \xi_{k-1}) \neq O$ since ξ_1, \dots, ξ_{k-1} are linearly independent. A contradiction has been reached. \square

End of the proof of Theorem 1.6. Let x_1, \dots, x_k be independent vectors of V , with $x_r = \sum_{i=1}^N e_i x_{i,r}$ for $r = 1, 2, \dots, k$. Put $x_r^\sigma := \sum_{j=1}^N x_{r,j}^\sigma e_j^* \in V^*$ and let X and X^σ be the tensors corresponding to $x_1 \wedge \dots \wedge x_k$ and $x_1^\sigma \wedge \dots \wedge x_k^\sigma$ respectively. Moreover, let Ξ be the tensor corresponding to $x^\sigma M_f \wedge \dots \wedge x^\sigma M_f$,

where M_f is the matrix representing f with respect to the basis $(e_i)_{i=1}^N$ of V . Then $\Xi = X^\sigma \circ^k (\odot^k M_f)$.

We have $x_r^\perp = \text{Ker}(x_r^\sigma M_f)$ for $r = 1, 2, \dots, k$, where \perp is the orthogonality relation associated to f . Therefore $\langle x_1, \dots, x_k \rangle$ is totally isotropic if and only if $\langle x_1, \dots, x_k \rangle \subseteq \cap_{i=1}^k \text{Ker}(x_i^\sigma M_f)$. By Lemma 2.2, this inclusion is equivalent to the relation $\Xi \circ X = O$, namely $X^\sigma \circ^k (\odot^k M_f) \circ X = O$. The proof is complete. \square

References

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